

Non-Existence of phase-shift breathers in one-dimensional Klein-Gordon lattices with nearest-neighbor interactions

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A one-dimensional Klein-Gordon lattice with nearest-neighbor interactions is studied. It is well known that this kind of systems can support multibreathers with phase differences between the successive oscillators $\phi_i = 0, \pi$. In this paper we prove the non-existence of *phase-shift breathers* i.e. multibreathers with $\phi_i \neq 0, \pi$ in this kind of systems. This fact also determines the linear stability of the existing configurations.

Keywords: Discrete Breathers, Multibreathers, Phase-shift breathers, Phase-shift multibreathers

I. INTRODUCTION

Since [20, 26] much interest has been drawn in the study of space-localized and time-periodic motions in lattices of coupled oscillators. These motions are called discrete breathers if the oscillation is localized around one “central” lattice site, while, if there are more than one central oscillators, the motion is called multibreather or multi-site breather.

Some of the applications of these localized modes lie in the fields of DNA double-strand dynamics in biophysics [22], coupled waveguide arrays and photorefractive crystals in non-linear optics [4, 8, 15], breathing oscillations in micromechanical cantilever arrays [24], localized modes in dusty plasma crystals [12, 13] Bose-Einstein condensates in optical lattices in atomic physics [19], and granular crystals [25]. The wide interest about discrete breathers- multibreathers is underlined by the numerous review papers there exist on this subject (e.g. [3, 6, 7, 16]).

One of the systems in which such motions are studied is the so called Klein-Gordon (KG) chain. This is a one-dimensional lattice of oscillators each one possessing a nonlinear on-site potential and being coupled with its nearest-neighbors with linear forces. The existing multibreathers are categorized in terms of the phase differences between the central oscillators. It is well known that KG chains can support multibreathers with phase differences between the successive central oscillators $\phi_i = 0, \pi$. These are the standard configurations. Although there were strong evidence that there cannot exist phase-shift breathers, i.e. multibreathers with phase differences $\phi_i \neq 0, \pi$, a rigorous proof of this fact was not yet achieved. The term phase-shift breathers is used for simplicity, instead of the term phase-shift multibreathers, in order to describe these motions, since there can be no single-site phase-shift breathers.

Since the first proof of existence of discrete breathers [17] there has been several papers dealing with the issue of existence and stability of multibreathers in KG chains. In [14] a methodology for proving the existence of multi-site breathers was introduced based in the work of [1, 18] and using also the terminology of [9]. This methodology was generalized for a generic Klein-Gordon chain in [10] and provided general persistence conditions independently of the precise form of the on-site potential. These results have been generalized in [21] by considering “holes” between the central oscillators and in [27] by considering an FPU chain.

The fact of the non-existence of phase-shift breathers in KG chains determines also the stability of the standard configurations. Specifically in [10] a general theorem about the stability of multibreathers is proven based on the assumption of the non-existence of phase-shift breathers, which is equivalent with the results of [2], as it has been recently shown in [5].

In this work we prove that the one-dimensional Klein-Gordon lattice with nearest-

neighbor interactions cannot support phase-shift breathers, by proving that the persistence conditions provided by [10] do not have solutions other than the standard ones $\phi_i = 0, \pi$. On the other hand, as it has been recently shown [11], in Klein-Gordon chains with interactions longer than the nearest-neighbor ones, phase-shift breathers can be supported and this fact dramatically changes also the stability picture for this kind of motions.

The paper is organized as follows. In section II we present briefly the methodology for the existence and stability of multibreathers in KG chains developed in [10], while we introduce some terminology. In section III the main theorem about the non-existence of phase-shift breathers is proven.

II. PERSISTENCE AND STABILITY OF MULTIBREATHERS IN 1D KLEIN-GORDON LATTICES WITH NEAREST-NEIGHBOR INTERACTIONS

In this section we will shortly present the main results of [10]. The classical Klein-Gordon (KG) setting is defined as a 1D chain of coupled oscillators each one moving on a nonlinear potential $V(x)$ possessing a local minimum at $x = 0$ ($V'(0) = 0, V''(0) = \omega_p^2 > 0$). Each oscillator is coupled with its two nearest neighbors with a linear coupling force through a coupling constant ε (Fig.1). The Hamiltonian of a Klein-Gordon chain with nearest neighbor

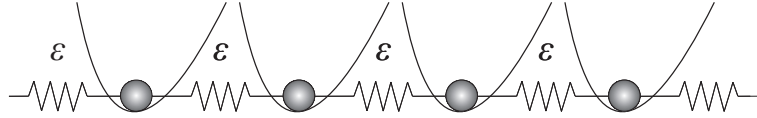


FIG. 1: A one-dimensional Klein-Gordon lattice with nearest-neighbor interactions

interactions is the following

$$H = H_0 + \varepsilon H_1 = \sum_{i=-\infty}^{\infty} \left[\frac{1}{2} p_i^2 + V(x_i) \right] + \frac{\varepsilon}{2} \sum_{i=-\infty}^{\infty} (x_i - x_{i-1})^2, \quad (1)$$

which leads to the equations of motion

$$\ddot{x}_i = -V'(x_i) + \varepsilon(x_{i-1} - 2x_i + x_{i+1}).$$

This system is well known to support discrete breather, as well as, multibreather solutions.

The key notion to the proof of the existence of multibreathers is the notion of the anti-continuum limit. This is the limit $\varepsilon \rightarrow 0$ where the chain consists of uncoupled oscillators. In this limit we consider all the oscillators of the chain at rest except for $n + 1$ adjacent “central” ones which move in periodic orbits of frequency ω , but with arbitrary phases. This configuration defines a trivially space-localized and time-periodic motion. But, not all of these configurations survive for $\varepsilon \neq 0$. In order for these motions to persist for $\varepsilon \neq 0$ to provide multibreathers, specific conditions on the phase differences between the oscillators must be satisfied, as well as, some rather generic non-degeneracy conditions.

In [1] it was shown that multibreathers correspond to critical points of H^{eff} which in first order of approximation is given by $H^{\text{eff}} = H_0(I_i) + \langle H_1 \rangle(\phi_i, I_i)$ [14]. The variables $\phi_i = w_{i+1} - w_i$ denote the n phase differences of the $n + 1$ successive central oscillators, while I_i are given by $I_i = \sum_{j=i}^n J_j$, where (J_i, w_i) are the action-angle variables of a single oscillator.

The average value of the coupling part of the Hamiltonian

$$\langle H_1 \rangle(\phi_i, I_i) = \frac{1}{T} \oint H_1(w_0, \phi_i, I_i) dt$$

is calculated along the orbits in the anti-continuum limit $\varepsilon = 0$.

This yields the conclusion that the persistence conditions for the existence of $n + 1$ -site multibreathers are

$$\frac{\partial \langle H_1 \rangle}{\partial \phi_i} = 0, \quad i = 1 \dots n, \quad (2)$$

as far as two non-degeneracy conditions hold. The first one is the non-resonance condition of the breather frequency ω with the phonon frequency ω_p i.e. $\omega_p \neq k\omega$. The second condition is the anharmonicity condition $\frac{\partial \omega}{\partial J} \neq 0$ which implies that the oscillation frequency of a single oscillator depends on the oscillation amplitude. Note that the persistence conditions are the same for every lattice case where the Hamiltonian can be written in the form $H = H_0 + \varepsilon H_1$ with $\frac{\partial \langle H_1 \rangle}{\partial \phi_i} \neq 0$.

By using the fact that the motion of the central oscillators for $\varepsilon = 0$ can be described by

$$x_i = \sum_{m=0}^{\infty} A_m \cos(mw_i) \quad (3)$$

the average value of H_1 becomes ([10] appendix A)

$$\langle H_1 \rangle = -\frac{1}{2} \sum_{m=1}^{\infty} \sum_{s=1}^n A_m^2 \cos(m\phi_s)$$

and the persistence conditions (2) become in the case of Klein-Gordon chains with nearest neighbor interactions,

$$\frac{\partial \langle H_1 \rangle}{\partial \phi_i} = 0 \Rightarrow M(\phi) \equiv \sum_{m=1}^{\infty} mA_m^2 \sin(m\phi_i) = 0, \quad i = 1 \dots n. \quad (4)$$

The function $M(\phi)$ possesses the obvious solutions $\phi_i = 0, \pi$, while, as it will be shown in the next section it possesses no others.

In figure 2 all the possible standard 3-site breather configurations are shown. These are (a) the in-phase $\{\phi_1 = \phi_2 = 0\}$ configuration, (b) the anti-phase $\{\phi_1 = \phi_2 = \pi\}$ configuration and (c) the mixed one $\{\phi_1 = 0, \phi_2 = \pi\}$. One could expect the oscillators which have phase difference $\phi_i = 0$ to be exactly one next to the other. Though this is not true as it can be also seen in fig.2 since for $\varepsilon \neq 0$ the effect of the rest of the lattice come to play. This is true only in the anticontinuum limit $\varepsilon = 0$. The on-site potential which is used in order to acquire these figures is $V(x) = x^2/2 - 0.15x^3/3 - 0.05x^4/4$. The same potential will be used for all the numerical calculations throughout this work.

We have to note here that in a recent preprint [21] the authors considered configurations with holes between the excited oscillators in the anticontinuum limit, by using higher order perturbation theory. But, in this work we will only consider adjacent central oscillators.

The spectral stability of the above mentioned multibreather solutions or, equivalently, the linear stability of the corresponding periodic orbits is determined through its *characteristic exponents* σ_i . These exponents are connected with the corresponding Floquet multipliers by the relation

$$\lambda_i = e^{\sigma_i T},$$

where $T = 2\pi/\omega$ is the period of the multibreather. Due to the Hamiltonian character of the system there is a pair of exponents identically equal to zero, while the nonzero characteristic exponents are given to leading order of approximation by

$$\sigma_{\pm i} = \pm \sqrt{-\varepsilon \frac{\partial \omega}{\partial J} \chi_{zi}} \quad i = 1 \dots n, \quad (5)$$

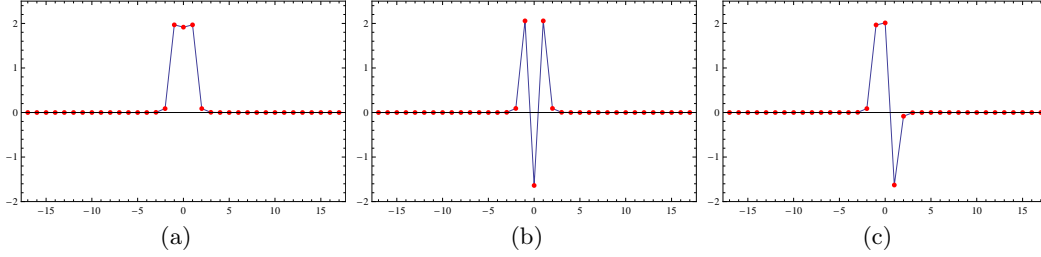


FIG. 2: Snapshots of all the possible 3-site breather configurations. In (a) the in-phase $\{\phi_1 = \phi_2 = 0\}$ configuration is shown, in (b) the anti-phase $\{\phi_1 = \phi_2 = \pi\}$ and in (c) the mixed one $\{\phi_1 = 0, \phi_2 = \pi\}$.

where χ_{z_i} are the eigenvalues of \mathbf{Z} with

$$\mathbf{Z} = \begin{pmatrix} 2f_1 & -f_1 & 0 & 0 \\ -f_2 & 2f_2 & -f_2 & 0 \\ & \ddots & \ddots & \ddots \\ 0 & -f_{n-1} & 2f_{n-1} & -f_{n-1} \\ & 0 & -f_n & 2f_n \end{pmatrix} \quad (6)$$

with

$$f_i = f(\phi_i) = \frac{1}{2} \sum_{m=1}^{\infty} m^2 A_m^2 \cos(m\phi_i). \quad (7)$$

Note that, for linear stability we require all the Floquet multipliers to lie on the unit circle, which is tantamount to all the characteristic exponents being purely imaginary. This depends on the sign of $P = \varepsilon \frac{\partial \omega}{\partial J}$ and the sign of χ_z as it can be seen from (5). Finally, we obtain the following theorem which determines the linear stability of the multibreather solutions in a 1D Klein-Gordon chain.

Theorem 1. *Under the assumption that (4) has only the $\phi_i = 0, \pi$ solutions, in systems of the form (1), if $P \equiv \varepsilon \frac{\partial \omega}{\partial J} < 0$ the only configuration which leads to linearly stable multibreathers, for $|\varepsilon|$ small enough, is the one with $\phi_i = \pi \quad \forall i = 1 \dots n$ (anti-phase multibreather), while if $P > 0$ the only linearly stable configuration, for $|\varepsilon|$ small enough, is the one with $\phi_i = 0 \quad \forall i = 1 \dots n$ (in-phase multibreather). Moreover, for $P < 0$ (respectively, $P > 0$), for unstable configurations, their number of unstable eigenvalues will be precisely equal to the number of nearest neighbors which are in- (respectively, in anti-) phase between them.*

The above theorem is proven under the assumption of non-existence of phase-shift breathers. The proof is based in the counting of the positive and negative eigenvalues of \mathbf{Z} which as it is shown in ([23] appendix C) it is directly connected with the sign of $f(0)$ and $f(\pi)$. The fact that $f(0) > 0$ is obvious from the definition in (7), while the proof of $f(\pi)$ ([10] lemma 3) requires the non-existence of phase shift breathers.

So, from what it is mentioned above, we can realize that the non-existence of phase-shift breathers is crucial, not only in order to exclude the non-supported multibreather configurations in 1D KG chains but also in order to categorize the supported ones in terms of their corresponding stability.

III. PROOF OF NONEXISTENCE OF PHASE-SHIFT BREATHERS IN 1D KLEIN-GORDON CHAINS WITH NEAREST-NEIGHBOR INTERACTIONS

In this section we will prove that phase-shift breathers cannot be supported in one-dimensional Klein-Gordon chains with nearest-neighbor interactions. Note that we use the term “phase-shift breathers” instead of the term “phase-shift breathers”. This is done for simplicity, since the phase-shift breathers are by definition multi-site because they are characterized by the phase difference between the central oscillators.

This will be proven by showing that the persistence conditions (4) have only the $\phi_i = 0, \pi$ solutions in the $\phi_i \in [0, 2\pi)$ interval.

In order to prove our main theorem we have first to prove some lemmas.

Lemma 1. *The solutions of $M(\phi) = 0$ coincide with the solutions of $I(\phi) \equiv \int_0^{2\pi} x(w)\dot{x}(w - \phi)dw = 0$.*

Proof. As we have already mentioned, the displacement of an uncoupled oscillator from the equilibrium can be described as an even 2π -periodic function with respect to $w = \omega t + \vartheta$, as

$$x(w) = \sum_{m=0}^{\infty} A_m \cos(mw). \quad (8)$$

We define the function $N(\phi)$ as the opposite of the averaged autocorrelation function of $x(w)$

$$N(\phi) = -\frac{1}{2\pi} \int_0^{2\pi} x(w)x(\phi - w)dw. \quad (9)$$

By substituting (8) into (9) we get

$$N(\phi) = -\frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_n A_m \int_0^{2\pi} \cos(nw) \cos[m(\phi - w)]dw$$

or

$$N(\phi) = -\frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_n A_m \int_0^{2\pi} \{\cos[(n - m)w + m\phi] + \cos[(n + m)w - m\phi]\}dw$$

which leads to

$$N(\phi) = -\frac{1}{2} \sum_{m=0}^{\infty} A_m^2 \cos(m\phi). \quad (10)$$

By differentiation of (10) with respect to ϕ we get

$$\frac{d}{d\phi} N(\phi) = \frac{1}{2} \sum_{m=0}^{\infty} m A_m^2 \sin(m\phi) = \frac{1}{2} M(\phi). \quad (11)$$

On the other hand, by using the differentiation properties of the convolution function, we get from (9),

$$\frac{d}{d\phi} N(\phi) = -\frac{1}{2\pi} \int_0^{2\pi} x(w) \frac{d}{dw} [x(\phi - w)]dw = \frac{1}{2\pi\omega} \int_0^{2\pi} x(w)\dot{x}(\phi - w)dw. \quad (12)$$

From (11) and (12) we get finally

$$M(\phi) \equiv \sum_{m=0}^{\infty} mA_m^2 \sin(m\phi) = \frac{1}{\pi\omega} \int_0^{2\pi} x(w)\dot{x}(\phi-w)dw = -\frac{1}{\pi\omega} \int_0^{2\pi} x(w)\dot{x}(w-\phi)dw. \quad (13)$$

So,

$$M(\phi) = 0 \Leftrightarrow I(\phi) \equiv \int_0^{2\pi} x(w)\dot{x}(w-\phi)dw = 0$$

□

Lemma 2. *The value of $I(\phi)$ for $\phi = 0$ or π is zero; $I(0) = I(\pi) = 0$.*

Proof. Let $\phi = 0$

As it can be seen in fig.3, both the displacement function $x(w)$ and the corresponding velocity $\dot{x}(w)$ possess some symmetries due to the Hamiltonian character of the oscillation. More precisely, $x(w)$ satisfies the symmetries $x(w) = x(-w)$ as well as $x(\pi-w) = x(\pi+w)$ while for $\dot{x}(w)$ we have $\dot{x}(w) = -\dot{x}(-w)$ as well as $\dot{x}(\pi-w) = -\dot{x}(\pi+w)$. On the other hand, $x(w)$ changes sign at $w = T_1$ and at $w = 2\pi - T_1$ ($T_1 \neq \pi/2$ in general due to the nonlinearity of the oscillator), while $p(w)$ changes sign at $w = 0$ (or 2π) and $w = \pi$. In addition, $x(w)$ possesses a local maximum at $w = 0, 2\pi$ and a local minimum at $w = \pi$, while $\dot{x}(w)$ possesses a local minimum at $w = T_1$ and a local maximum at $w = 2\pi - T_1$. Both functions are strictly monotonic between their local extrema.

So, the integral $I(0)$ can be divided in four qualitatively different regions as

$$\begin{aligned} I(0) \equiv \int_0^{2\pi} x(w)\dot{x}(w)dw &= \int_0^{T_1} x(w)\dot{x}(w)dw + \int_{T_1}^{\pi} x(w)\dot{x}(w)dw + \\ &+ \int_{\pi}^{2\pi-T_1} x(w)\dot{x}(w)dw + \int_{2\pi-T_1}^{2\pi} x(w)\dot{x}(w)dw \end{aligned}$$

or

$$I(0) = I_1 + I_2 + I_3 + I_4, \quad (14)$$

where $I_1, I_3 < 0$ and $I_2, I_4 > 0$, as it can be seen in fig.3.

On the other hand, by using the symmetries mentioned earlier, we have

$$x(z) = x(-z) = x(2\pi - z) \quad \text{and} \quad \dot{x}(z) = -\dot{x}(-z) = -\dot{x}(2\pi - z)$$

which gives, by multiplication by parts,

$$x(z)\dot{x}(z) = -x(2\pi - z)\dot{x}(2\pi - z) \Rightarrow \int_0^{T_1} x(z)\dot{x}(z)dz = -\int_0^{T_1} x(2\pi - z)\dot{x}(2\pi - z)dz$$

or, by substituting $v_1 = z$ to the lhs of the above equation and $v_2 = \pi - z$ to the rhs, we get

$$\int_0^{T_1} x(v_1)\dot{x}(v_1)dv_1 = -\int_{2\pi-T_1}^{2\pi} x(v_2)\dot{x}(v_2)dv_2$$

and since v_1 and v_2 are dummy variables, we finally get

$$\int_0^{T_1} x(w)\dot{x}(w)dw = -\int_{2\pi-T_1}^{2\pi} x(w)\dot{x}(w)dw \Rightarrow I_1 = -I_4. \quad (15)$$

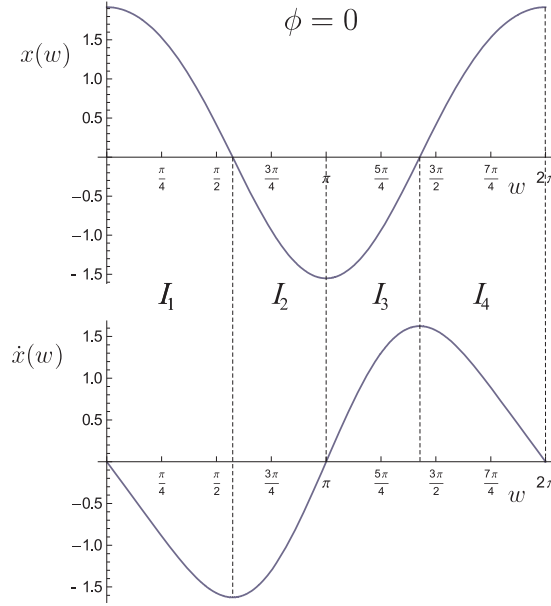


FIG. 3: The displacement $x(w)$ and the velocity $\dot{x}(w)$ of an uncoupled oscillator in the $\phi = 0$ case. In the figure, the different integration intervals are depicted.

On the other hand, he have

$$x(\pi + z) = x(\pi - z) \quad \text{and} \quad \dot{x}(\pi + z) = -\dot{x}(\pi - z),$$

which gives

$$x(\pi + z)\dot{x}(\pi + z) = -x(\pi - z)\dot{x}(\pi - z) \Rightarrow$$

$$\int_0^{\pi-T_1} x(z+\pi)\dot{x}(z+\pi)dz = -\int_0^{\pi-T_1} x(\pi-z)\dot{x}(\pi-z)dz$$

or, by the proper change of variables,

$$\int_{\pi}^{2\pi-T_1} x(w)\dot{x}(w)dw = -\int_{T_1}^{\pi} x(w)\dot{x}(w)dw \Rightarrow I_2 = -I_3. \quad (16)$$

So, by eqs. (14), (15) and (16), we finally get

$$I(0) = I_1 + I_2 + I_3 + I_4 = 0 \Rightarrow M(0) = 0.$$

For $\phi = \pi$,
it is

$$I(\pi) = \int_0^{2\pi} x(w)\dot{x}(w-\pi)dw = -\int_0^{2\pi} x(w)\dot{x}(w)dw = -I(0) = 0.$$

□

Remark: The above proof is somehow “unnecessary” and could be done, a lot easier, in two ways. Firstly, since

$$I(\phi) \equiv \int_0^{2\pi} x(w)\dot{x}(w-\phi)dw = -2\pi\omega \sum_{n=0}^{\infty} mA_m^2 \sin(m\phi)$$

it is $I(\phi) = 0$ for $\phi = 0, \pi$ by construction. Secondly, due to the parity property of the product $x(w)\dot{x}(w)$ (or equivalently the $x(w)\dot{x}(w - \pi)$ product), the corresponding integral over a period is zero.

But we chose to present the above proof as a guideline for the proof of lemma 3.

Lemma 3. *The value of $I(\phi)$ for $\phi \in (0, 2\pi) \setminus \{\pi\}$ is different from zero. In particular $I(\phi) > 0$ for $\phi \in (0, \pi)$ and $I(\phi) < 0$ for $\phi \in (\pi, 2\pi)$.*

Proof. Let $0 < \phi < \pi$.

We will examine first the case example of $\phi = \pi/8$. Since it is

$$\int_0^{2\pi} x(w)\dot{x}(w - \phi)dw = \int_\phi^{2\pi+\phi} x(w)\dot{x}(w - \phi)dw,$$

due to the periodicity of the functions under integration, we choose to use the rhs integral instead of the original one, since it is more convenient for calculations. In this case, we divide the integral under consideration into six parts, as shown in fig.4.

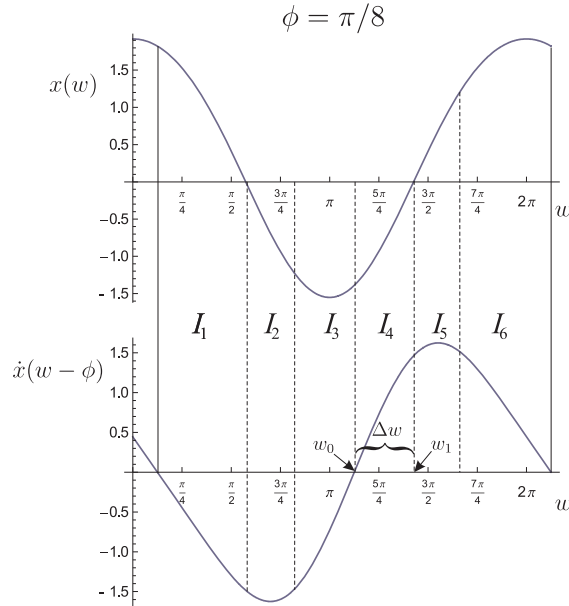


FIG. 4: The displacement $x(w)$ and the phase-shifted velocity $\dot{x}(w - \phi)$ of an uncoupled oscillator in the $\phi = \pi/8$ case. In the figure, the different integration intervals are depicted.

Our goal is to show that $I_1 + I_6 > 0$, $I_3 + I_4 > 0$ and since it is $I_2, I_5 > 0$, we get finally

$$I\left(\frac{\pi}{8}\right) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 > 0.$$

We will show first that $I_3 + I_4 > 0$. Let $w_0 = \pi + \phi$ and $\Delta w = w_1 - w_0 = 2\pi - T - (\pi + \phi) = \pi - T_1 - \phi$. Then for $0 \leq z \leq \Delta w$ it is

$$-\dot{x}(\pi - z) = \dot{x}(\pi + z) \Rightarrow -\dot{x}(w_0 - \phi - z) = \dot{x}(w_0 - \phi + z) > 0 \quad (17)$$

and

$$x(w_0 - z) < x(w_0 + z). \quad (18)$$

The last inequality is obvious in the monotonic part of $x(w)$ ($\pi \leq w_0 - z \leq w_1 = 2\pi - T_1 \Rightarrow z \leq \phi$). But this is true even when $w_0 - z < \pi \Rightarrow z > \phi$. Let $z = \phi + \Delta z = w_0 - \pi + \Delta z$, then, by using the $x(\pi + z) = x(\pi - z)$ symmetry, we have

$$x(w_0 - z) = x(w_0 - (w_0 - \pi + \Delta z)) = x(\pi - \Delta z) = x(\pi + \Delta z) = x(2\pi - w_0 + z) < x(w_0 + z)$$

in the last inequality we used the fact that $2\pi - w_0 + z = \pi - \phi + z$ and $\pi < \pi + \Delta z = \pi - \phi + z < \pi + \phi + z = w_0 + z$.

So, from eqs. (17, 18) we get, for $0 \leq z \leq \Delta w$,

$$\begin{aligned} & -x(w_0 - z) \dot{x}(w_0 - \phi - z) < x(w_0 + z) \dot{x}(w_0 - \phi + z) \\ & - \int_0^{\Delta w} x(w_0 - z) \dot{x}(w_0 - \phi - z) dz < \int_0^{\Delta w} x(w_0 + z) \dot{x}(w_0 - \phi + z) dz \end{aligned}$$

we perform the $z = w_0 - v_1$ change of variables to the lhs and the $z = w_0 + v_2$ one to the rhs and we get

$$\int_{w_0}^{w_0 - \Delta w} x(v_1) \dot{x}(v_1 - z) dv_1 < \int_{w_0}^{w_0 + \Delta w} x(v_2) \dot{x}(v_2 - \phi) dv_2$$

or, since v_1, v_2 are dummy variables

$$\begin{aligned} & - \int_{w_0 - \Delta w}^{w_0} x(w) \dot{x}(w - \phi) dw < \int_{w_0}^{w_0 + \Delta w} x(w) \dot{x}(w - \phi) dw \\ & \int_{w_0 - \Delta w}^{w_0} x(w) \dot{x}(w - \phi) dw + \int_{w_0}^{w_0 + \Delta w} x(w) \dot{x}(w - \phi) dw > 0 \\ & \int_{2\phi + T_1}^{\pi + \phi} x(w) \dot{x}(w - \phi) dw + \int_{\pi + \phi}^{2\pi - T_1} x(w) \dot{x}(w - \phi) dw > 0 \end{aligned}$$

$$I_3 + I_4 > 0. \tag{19}$$

We will consider now the $I_1 + I_6$ sum. Let $w_0 = \phi$ and $\Delta w = T_1 - \phi$. Then we have for $0 \leq z \leq \Delta w$

$$-\dot{x}(z) = \dot{x}(2\pi - z) > 0$$

and

$$x(\phi + z) < x(2\pi + \phi - z),$$

where we have used similar symmetry arguments as the ones we used in order to acquire inequality (18). So,

$$\begin{aligned} & -x(\phi + z) \dot{x}(z) < x(2\pi + \phi - z) \dot{x}(2\pi - z) \\ & - \int_0^{\Delta w} x(\phi + z) \dot{x}(z) dz < \int_0^{\Delta w} x(2\pi + \phi - z) \dot{x}(2\pi - z) dz \end{aligned}$$

Let $z = v_1 - \phi$ for the lhs and $2\pi - z = v_2 - \phi$ for the rhs

$$\begin{aligned}
& - \int_{\phi}^{\phi+\Delta w} x(v_1) \dot{x}(v_1 - \phi) dv_1 < \int_{2\pi+\phi}^{2\pi+\phi-\Delta w} x(v_2) \dot{x}(v_2 - \phi) dv_2 \\
& \int_{\phi}^{\phi+\Delta w} x(w) \dot{x}(w - \phi) dw + \int_{2\pi+\phi-\Delta w}^{2\pi+\phi} x(w) \dot{x}(w) dw > 0 \\
& I_1 + I_6 > 0
\end{aligned} \tag{20}$$

Since it is also $I_2, I_5 > 0$, we get also from (19) and (20) that

$$I(\pi/8) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 > 0.$$

As it is obvious from the proof procedure above, in order to prove that

$$I(\phi) = \int_0^{2\pi} x(w) \dot{x}(w - \phi) dw > 0 \quad \forall 0 < \phi < \pi$$

we have to examine separately various intervals of ϕ depending on the behavior of $x(w)$ and $\dot{x}(w - \phi)$ in these regions. In order to determine these regions we have first to determine whether $T_1 > \frac{\pi}{2}$ or $T_1 < \frac{\pi}{2}$. Without loss of generality, we will consider the first case since that is the case for the on-site potential we use in this work. For the $T_1 < \frac{\pi}{2}$ case, the methodology is similar. The qualitatively different intervals of ϕ , we have to examine in the $(0, \pi)$ interval, are:

- $0 < 2\phi \leq \pi - T_1$: This is the case for $\phi = \pi/8$ which we have already examined and is depicted in fig.4. In this region the integration interval of I_3 and I_6 contain the extrema of $x(w)$ so we have to use the symmetry properties around the extrema as we did in order to prove inequality (18).
- $\pi - T_1 < 2\phi \leq T_1$: As an example of this interval we consider $\phi = \pi/4$, which is depicted together with all the remaining cases in fig.5. In this case again we calculate the $I(\phi) = \int_{\phi}^{2\pi+\phi} x(w) \dot{x}(w - \phi) dw$ integral which is divided in six regions. Only the integration interval of I_6 contains an extremum of $x(w)$ and we need the symmetry properties around the extremum like the ones we used in (18) in order to prove $I_1 + I_6 > 0$. In addition we have $I_3 + I_4 > 0$ and $I_2, I_5 > 0$, so finally $I(\phi) > 0$.
- $T_1 < 2\phi \leq 2\pi - T_1$: As an example of this interval we consider $\phi = 3\pi/8$. We use $[0, 2\pi]$ as the integration interval for the calculation of $I(\phi)$, which is divided in 7 regions, as it shown in fig. 5. In this case we can show, by using similar arguments as in the previous cases, that $I_2 + I_3 > 0$ and $I_5 + I_6 > 0$ while $I_1, I_2, I_3 > 0$. So, finally $I(\phi) > 0$. Note that in this case none of the I_2, I_3, I_5, I_6 contains an extremum of $x(w)$ so no extra symmetry conditions around the extrema are needed for the argument to be proved.
- $2\pi - T_1 < 2\phi \leq \pi + T_1$: As an example of this interval we consider $\phi = 3\pi/4$. In this case we use the integration interval $[\pi/4, 9\pi/4]$ for the calculation of $I(\phi)$ which we divide in seven regions, as it can be seen in fig. 5. In this case we have $I_2 + I_3 > 0$, $I_5 + I_6 > 0$, while $I_1, I_4, I_7 > 0$, so, $I(\phi) > 0$.
- $\pi + T_1 < 2\phi < \pi$: As an example of this interval we consider $\phi = 7\pi/8$. In this case we use the integration interval $[\pi/2, 5\pi/2]$ for the calculation of $I(\phi)$ which we divide again in seven regions, as it can be seen in fig. 5. In this case we have $I_2 + I_3 > 0$, $I_5 + I_6 > 0$, while $I_1, I_4, I_7 > 0$, so, $I(\phi) > 0$.

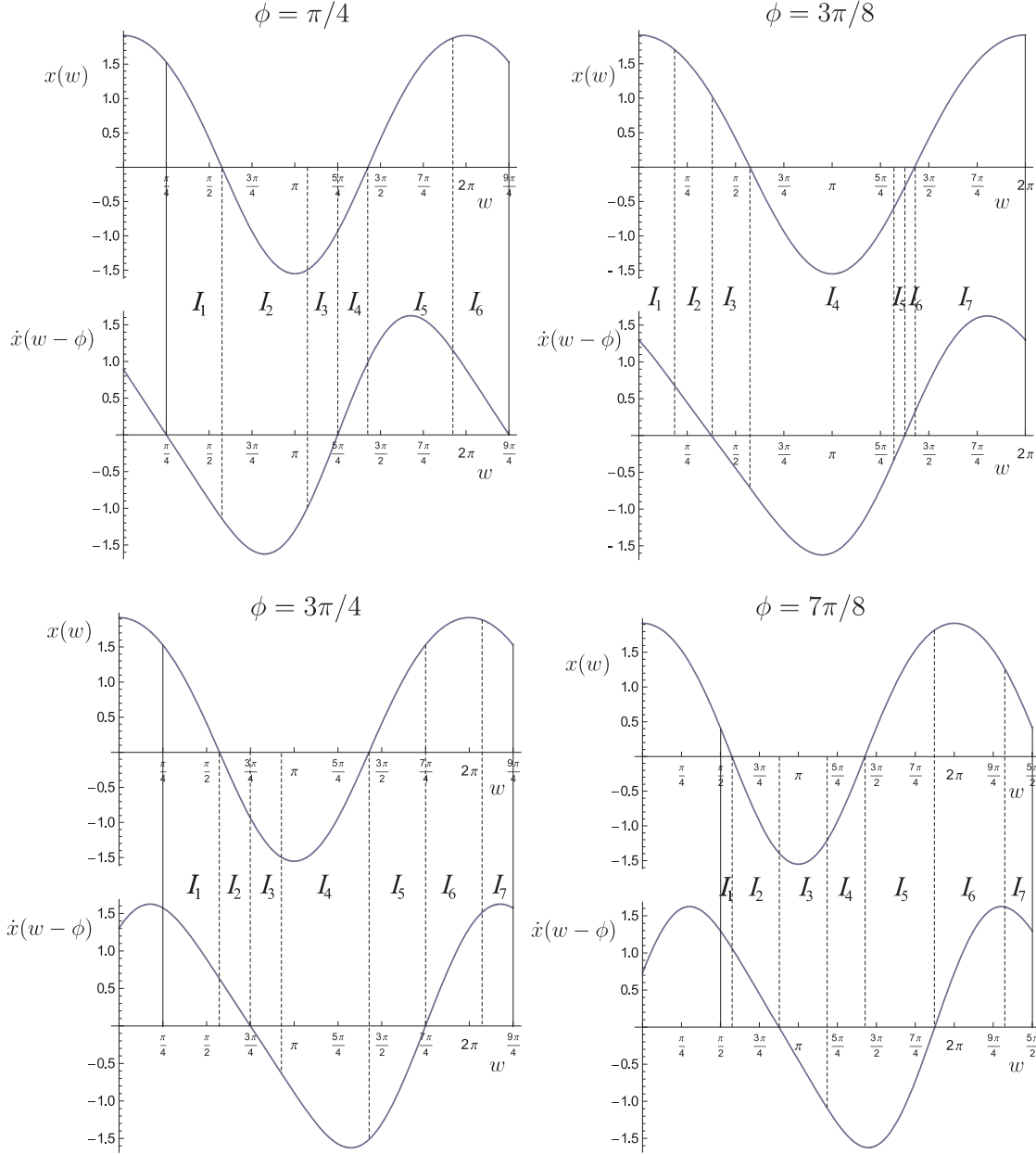


FIG. 5: The displacement $x(w)$ and the phase-shifted velocity $\dot{x}(w - \phi)$ of an uncoupled oscillator in the $\phi = \pi/4$, $\phi = 3\pi/8$, $\phi = 3\pi/4$, $\phi = 7\pi/8$ cases. The solid lines, which are vertical to the axes, represent the integration interval we use for the calculation of $I(\phi)$ in the various cases. The subintervals we use for this calculation in each case are depicted as well.

So, since we have shown that our claim holds for ϕ in every subinterval $0 < \phi < \pi$ we conclude that

$$I(\phi) > 0, \quad \forall \phi \in (0, \pi).$$

With exactly similar arguments it can be shown that

$$I(\phi) < 0, \quad \forall \phi \in (\pi, 2\pi).$$

□

Now we are ready to proof our main theorem.

Theorem 2. *In one-dimensional KG with nearest neighbor interactions no phase-shift multibreathers can be supported.*

Proof. In order to prove this theorem it is sufficient to show that the persistence conditions (4) have only the $\phi_i = 0, \pi \bmod 2\pi$ solutions $\forall i = 1 \dots n$. Since eqs. (4) consists of n independent equations, we have only to examine just one of them i.e. $M(\phi) = 0$. Indeed, since by lemmas 2 and 3 we know that for $\phi \in [0, 2\pi)$ the only solutions of $I(\phi) = 0$ are $\phi = 0$ and π this means, by lemma 1, that $M(\phi) = 0$ has the same solutions. □

IV. CONCLUSIONS

It is well known that one-dimensional Klein-Gordon (KG) lattices with nearest-neighbor (NN) interactions support multibreathers with the standard phase-difference $\phi_i = 0, \pi$ between adjacent central oscillators. On the other hand there are strong evidences that phase-shift breathers i.e. multibreathers with $\phi_i \neq 0$ or π cannot exist in this classical KG setting.

In the present work we prove that, indeed, the only configurations that can exist in a classical KG 1D lattice with NN interactions are the standard one $\phi_i = 0, \pi$. This fact excludes the existence of phase-shift breathers and, as it has been shown in [10], it also clarifies the stability image for the existing multibreathers i.e. if $P \equiv \varepsilon \frac{\partial \omega}{\partial J} < 0$ the anti-phase configuration is the only stable one, while for $P > 0$ the in-phase configuration is the only stable multibreather solution.

On the other hand, as it has been recently shown [11], in 1D KG chains where interactions with range larger than just the nearest-neighbor ones are considered, phase-shift breathers can be supported, giving rise to radically different stability scenaria.

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- [1] T. Ahn, R. S. MacKay, and J.-A. Sepulchre. Dynamics of relative phases: Generalised multibreathers. *Nonlinear Dynamics*, 25:157–182, 2001.
 - [2] J. F. R. Archilla, J. Cuevas, B. Sánchez-Rey, and A. Álvarez. Demonstration of the stability or instability of multibreathers at low coupling. *Physica D*, 180:235, 2003.
 - [3] S. Aubry. Breathers in nonlinear lattices: existence, linear stability and quantization. *Physica D*, 103:201–250, 1997.
 - [4] G. Bartal, O. Cohen, T. Schwartz, O. Manela, Fredman B., M. Segev, H. Buljan, and N. K. Efremidis. Spatial photonics in nonlinear waveguide arrays. *Opt. Express*, 13:1780, 2005.
 - [5] J. Cuevas, V. Koukoulouyannis, P. G. Kevrekidis, and J. F. R. Archilla. Multibreather and vortex breather stability in Klein–Gordon lattices: Equivalence between two different approaches. *Int. J. Bifur. Chaos*, 21:2161, 2011.

- [6] S. Flach and A. V. Gorbach. Discrete breathers - Advances in theory and applications. *Phys. Rep.*, 467:1, 2008.
- [7] S. Flach and C. R. Willis. Discrete breathers. *Phys. Rep.*, 295:182, 1998.
- [8] Yu. S. Kivshar and G. P. Agrawal. *Optical solitons: from fibers to photonic crystals*. Academic Press, San Diego, 2003.
- [9] V. Koukouloyannis and S. Ichtiaroglou. Existence of multibreathers in chains of coupled one-dimensional hamiltonian oscillators. *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics*, 66(6):066602, 2002.
- [10] V. Koukouloyannis and P. G. Kevrekidis. On the stability of multibreathers in Klein–Gordon chains. *Nonlinearity*, 22:2269, 2009.
- [11] V. Koukouloyannis, P. G. Kevrekidis, J. Cuevas, and V. Rothos. Multibreathers in klein-gordon chains beyond nearest neighbors. *ArXiv:1204.5496*, 2012.
- [12] V. Koukouloyannis and I. Kourakis. Existence of multisite intrinsic localized modes in one-dimensional debye crystals. *Phys. Rev. E*, 76:016402, 2007.
- [13] V. Koukouloyannis and I. Kourakis. Discrete breathers in hexagonal dusty plasma lattices. *Phys. Rev. E*, 80:026402, 2009.
- [14] V. Koukouloyannis and R. S. MacKay. Existence and stability of 3-site breathers in a triangular lattice. *J. Phys. A: Math. Gen.*, 38:1021, 2005.
- [15] F. Lederer, G. I. Stegeman, D.N. Christodoulides, G. Assanto, M. Segev, and Y. Silberberg. Discrete solitons in optics. *Phys. Rep.*, 463:1, 2008.
- [16] R. S. MacKay. Discrete breathers: classical and quantum. *Physica A*, 288:174–198, 2000.
- [17] R. S. MacKay and S. Aubry. Proof of the existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators. *Nonlinearity*, 7:1623, 1994.
- [18] R. S. MacKay and J.-A. Sepulchre. Effective Hamiltonian for traveling discrete breathers. *J. Phys. A: Math. Gen.*, 35:3985, 2002.
- [19] O. Morsch and M. O. Oberthaler. Dynamics of bose-einstein condensates in optical lattices. *Rev. Mod. Phys.*, 78:179, 2006.
- [20] J. B. Page. Asymptotic solutions for localized vibrational modes in strongly anharmonic periodic systems. *Phys. Rev. B*, 41:7835, 1990.
- [21] D. E. Pelinovsky and A. Sakovich. Multi-site breathers in Klein-Gordon lattices: stability, resonances and bifurcations. *ArXiv:1111.2557*, 2011.
- [22] M. Peyrard. Nonlinear dynamics and statistical mechanics of DNA. *Nonlinearity*, 17:R1, 2004.
- [23] B. Sandstede. Stability of multiple-pulse solutions. *Trans. Am. Math. Soc.*, 350:429, 1998.
- [24] M. Sato, B. E. Hubbard, and A. J. Sievers. Nonlinear energy localization and its manipulation in micromechanical oscillator arrays. *Rev. Mod. Phys.*, 78:137, 2006.
- [25] S. Sen, J. Hong, J. Bang, E. Avalos, and E. Doney. Solitary waves in the granular chain. *Phys. Rep.*, 462:21, 2008.
- [26] A. J. Sievers and S. Takeno. Intrinsic localized modes in anharmonic crystals. *Physical Review Letters*, 61:970–973, 1988.
- [27] K. Yoshimura. Existence and stability of discrete breathers in diatomic fermi-pasta-ulam type lattices. *Nonlinearity*, 24:293, 2011.